NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

(NASA-TM-103458) SUMMARY AND COMPILATION OF IMPORTANT COMPRESSIBLE FLOW RELATIONS (NACA) 59 p

N90-71308

Unclas
00/02 0293167

SUMMARY AND COMPILATION OF IMPORTANT COMPRESSIBLE FLOW RELATIONS

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April 1944

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CONTENTS

INDEX

Notation and basic relations

ρ	mass density of fluid
V	specific volume $= \frac{1}{\rho}$
p	static pressure
R	gas constant (defined for unit mass)
T	absolute temperature
C_p	specific heat at constant pressure (for unit mass)
C_v	specific heat at constant volume (for unit mass)
h	enthalpy $= C_pT$ (defined for unit mass)
ΔS_T	change in entropy $= \int_{T_1}^{T_2} \frac{dQ}{T}$
m	mass
J	mechanical equivalent of heat
γ	ratio of specific heats at constant pressure and constant volume
E	modulus of elasticity
Q	added heat energy per unit mass
W	work done by external forces on unit mass
x,y	rectangular coordinates
i =	$\sqrt{-1}$
z	complex variable $= x + iy$
φ	velocity potential
ψ	stream function
u,v	components of fluid velocity parallel to rectangular axes of x and y (redefined for polar coordinates)

$$u = \frac{\partial\varphi}{\partial x};\ v = \frac{\partial\varphi}{\partial y}$$

$$\rho u = \rho_o \frac{\partial\psi}{\partial y};\ \rho v = -\rho_o \frac{\partial\psi}{\partial x}$$

w resultant fluid speed in two dimensions, $w^2 = u^2 + v^2$, also component of velocity in direction of z-axis in three dimensions

δ angle between resultant fluid velocity and x-axis $u = w \cos\delta;\ v = w \sin\delta$

w_m maximum speed which can be attained by fluid = speed when $p = o$, $\rho = o$, and $a = o$

τ nondimensional speed variable $\tau = \left(\frac{w}{w_m}\right)^2$ $0 \leq \tau \leq 1$

a local speed of sound

M Mach number $= \frac{w}{a}$

P pressure coefficient $= \frac{\Delta p}{q}$

S area normal to direction of flow

q dynamic pressure, $\frac{1}{2}\rho w^2$

g acceleration of gravity

t time

$$\varphi_u = \frac{\partial\varphi}{\partial u};\ \varphi_{xx} = \frac{\partial^2\varphi}{\partial x^2};\ \varphi_{xy} = \frac{\partial^2\varphi}{\partial x\partial y} \text{ etc.: also } \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}$$

Polar coordinates

r, θ polar coordinates of point

u, v velocity along and perpendicular to radius vector respectively $u = -\frac{\partial\varphi}{\partial r};\ v = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}$

φ velocity potential (defined by above equations for polar coordinates)

ω	angular velocity
α	angular acceleration

Subscripts

o	stagnation point
s	free stream
l	local point in flow
t	throat section (minimum cross section)
i	incompressible flow
m	maximum value of variable
cr	critical value of variable (defined in text)
1,2	any two values of the variable

No subscript, any point in the flow

II

Thermodynamic relations

1. General gas law

   $pV = RT$ (where R is defined for unit mass)

2. Heat transfer at constant pressure or volume (specific heats)

   (a) At constant volume: $dQ = C_v m dT$

   (b) At constant pressure: $dQ = C_p m dT = C_v m dT + \frac{p\,d(mv)}{J}$

3. Expressions involving specific heats and general gas law (enthalpy)

   (a) $C_p - C_v = \frac{R}{J}$

   (b) $\frac{C_p - C_v}{C_v} = \frac{R}{JC_v}$; $\frac{C_p}{C_v} = \gamma$; $C_v = \frac{R}{J(\gamma-1)}$

   (c) $C_p T = C_v T + \frac{pV}{J}$

   (d) $h = C_p T =$ enthalpy

   (e) $h = \frac{RT}{J(\gamma-1)} + \frac{pV}{J} = \frac{pV}{J(\gamma-1)} + \frac{pV}{J}$

   (f) $h = \frac{\gamma}{\gamma-1} \frac{p}{\rho J}$: or $mh = \frac{\gamma}{\gamma-1} \frac{mp}{\rho J}$ (for mass m; h = enthalpy for unit mass)

4. General energy equation

   (a) Equation in differential form

   $$mJC_v dT + mpdV + mdW + mJdQ + m\frac{dw^2}{2} = \text{constant}$$

   (b) Integrated expression for unit mass

$$JC_vT_1 + p_1V_1 + W + JQ + \frac{w_1^2}{2} = JC_vT_2 + p_2V_2 + \frac{w_2^2}{2}$$

or $$Jh_1 + W + JQ + \frac{w_1^2}{2} = Jh_2 + \frac{w_2^2}{2}$$

(c) Energy equation for adiabatic flow ($W = Q = 0$)

$$Jh_1 + \frac{w_1^2}{2} = Jh_2 + \frac{w_2^2}{2}$$

(d) Energy equation for adiabatic flow involving stagnation ($w_o = 0$)

$$Jh_o = Jh + \frac{w^2}{2}$$

or $$w^2 = 2J(h_o - h)$$

and $$w^2_{max} = 2Jh_o$$

$$w_t^2 = a_t^2 = 2J \frac{\gamma-1}{\gamma+1} h_o$$

also $$\frac{\gamma p_o V_o}{\gamma-1} = \frac{\gamma pV}{\gamma-1} + \frac{w^2}{2}$$

or $$\frac{\gamma}{\gamma-1} \frac{p_o}{\rho_o} = \frac{\gamma}{\gamma-1} \frac{p}{\rho} + \frac{w^2}{2}$$

5. Isentropic flow relations

$$\frac{T_1}{T_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma-1} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

6. Other useful relations

(a) Adiabatic temperature rise

$$\Delta T = \frac{w^2}{2JC_p}$$

(b) Intrinsic energy of unit mass = work done against external pressure when gas is expanded adiabatically to zero density:

$$\int_{\left(\frac{1}{\rho_o}\right)}^{\infty} p \, d\left(\frac{1}{\rho}\right) = \frac{1}{\gamma-1} \frac{p_o}{\rho_o}$$

III

Velocity of Sound in Air

1. Methods of derivation

(a) Derivation by momentum considerations in one dimension and elastic bar analogy yields

$$a = \sqrt{\frac{E}{\rho}}$$

(b) Reduction to the general wave equation in one dimension

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

defines $$a^2 = \frac{dp}{d\rho}$$

2. The various expressions for the velocity of sound

(a) Substitution of the bulk modulus

$E = \rho\frac{dp}{d\rho}$ in 1(a) leads to

$$\frac{dp}{d\rho} = a^2 \quad \text{as in 1(b)}$$

(b) For the adiabatic process $\frac{dp}{d\rho} = \frac{\gamma p}{\rho}$

and

$$a = \sqrt{\frac{\gamma p}{\rho}}$$

(c) For the isothermal process $\frac{dp}{d\rho} = \frac{p}{\rho}$

and

$$a = \sqrt{\frac{p}{\rho}}$$

(d) From 2(b) and the general gas law $pV = RT$

$$a = \sqrt{\gamma RT}$$

3. Expressions relating the velocity of sound at several points in the flow (adiabatic process).

(a) Equation relating any two points on a streamline

$$\frac{a_1}{a_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{2\gamma}}$$

(b) Equation relating local and stream points

$$\frac{a_l}{a_s} = \left(\frac{p_l}{p_s}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{\gamma}{2}PM_s^{\,2} + 1\right)^{\frac{\gamma-1}{2\gamma}} = \left[1 + \frac{\gamma-1}{2}M_s^{\,2}\left(1 - \frac{w_l^{\,2}}{w_s^{\,2}}\right)\right]^{\frac{1}{2}}$$

(c) Equation relating stagnation and stream points

$$\frac{a_o}{a_s} = \left(1 + \frac{\gamma-1}{2}M_s^{\,2}\right)^{\frac{1}{2}}$$

(d) Equation relating throat and stagnation points.
Combination of the condition for sonic velocity

$$\frac{p_t}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$ with the adiabatic relation and 2(b) yields :—

$$\frac{a_t}{a_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}$$

(e) Equation relating throat and stream points

$$\frac{a_t}{a_s} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\left(1 + \frac{\gamma-1}{2}M_s{}^2\right)^{\frac{1}{2}}$$

4. Velocity of sound in terms of the maximum attainable velocity

$$a_o = \frac{w_m}{\sqrt{\frac{2}{\gamma-1}}}$$

and

$$a_t = w_m\sqrt{\frac{\gamma-1}{\gamma+1}}$$

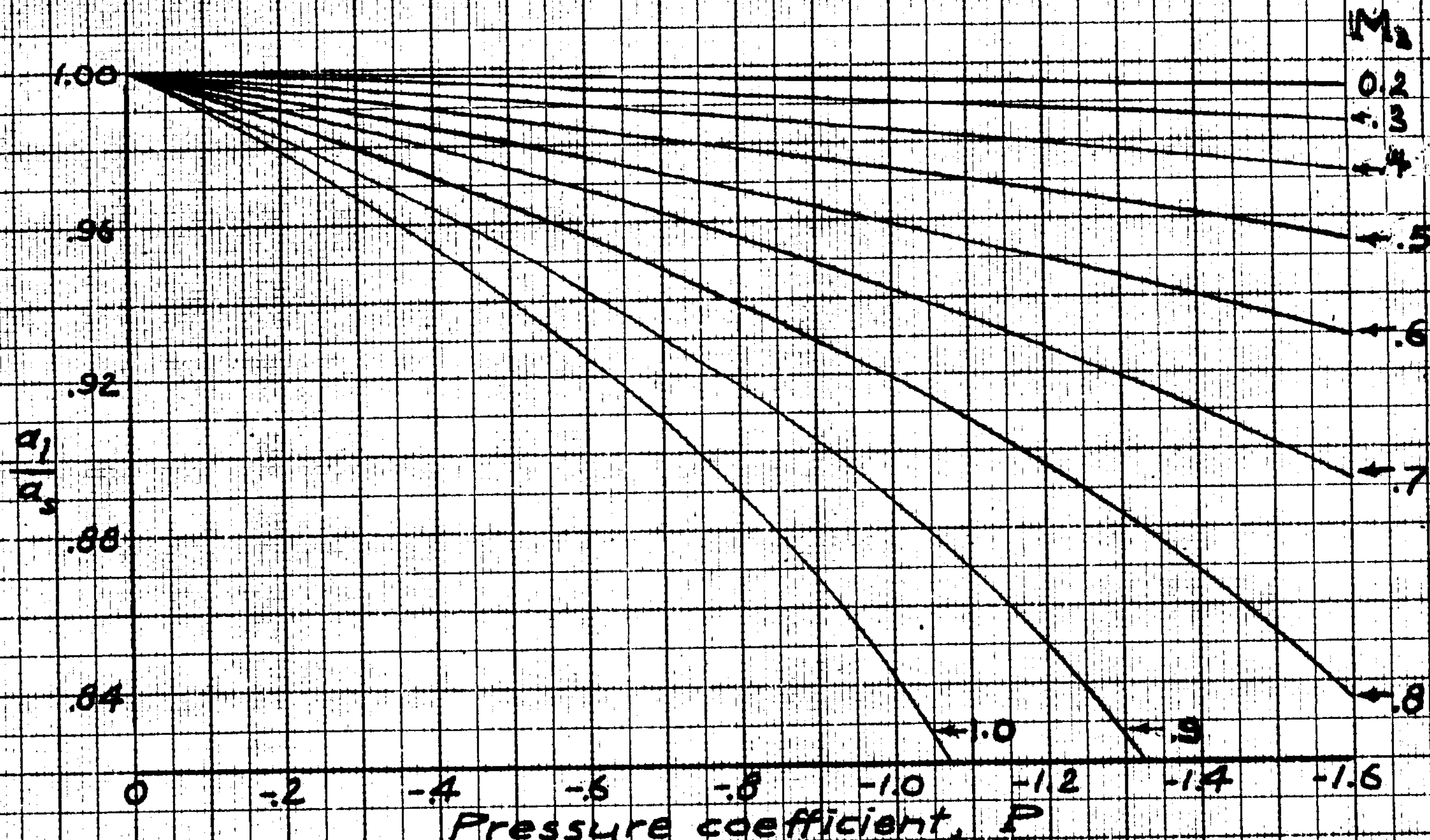


Figure III a.- Local speed of sound as a function of stream Mach number and local pressure coefficient.

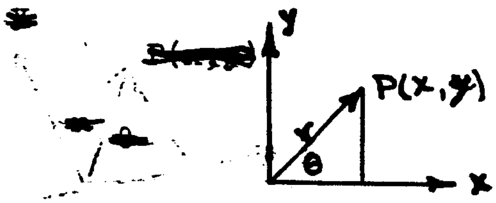
IV

Systems of Coordinates and Transformation Equations

1. Complex variable and potential function

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

the complex coordinate of $P(x,y)$ is given by



$z = re^{i\theta}$

The potential function $f(z) = \varphi + i\psi$ permits representation of any irrotational motion in one equation and has the property $\dfrac{df(z)}{dz} = u - iv$

2. The Joukowski transformation

The relation $\zeta = z + \dfrac{a^2}{z}$ maps the circle of radius a with its center at the origin of the z plane into the line segment $(-2a, 0; 2a, 0)$ in the ζ plane. The circles concentric with the circle of radius a are transformed into a family of confocal ellipses with common foci at $(-2a, 0)$ and $(2a, 0)$. If $R(>a)$ denotes the radius of one of these circles, then the semimajor and semiminor axes of the ellipse into which it is transformed are, respectively, $R + \dfrac{a^2}{R}$ and $R - \dfrac{a^2}{R}$; the thickness ratio becomes

$$t = \frac{R-\frac{a^2}{R}}{R+\frac{a^2}{R}} = \frac{1-\sigma^2}{1+\sigma^2}; \quad \sigma = \frac{a}{R}$$

By displacing the center of the circle, a whole series of airfoil shapes are obtained (Joukowski profiles) and the corresponding velocity and pressure distribution can be calculated.

3. (a) Legendre contact transformation for two independent variables

$$X = \varphi_x = u$$

$$Y = \varphi_y = v$$

$$\chi = \varphi_x x + \varphi_y y - \varphi = ux + vy - \varphi$$

The surface $\varphi = \varphi\,(x,y)$ goes over into the surface

$$\chi = \chi\,(X,Y)$$

(b) By means of the previous transformation, the equation for the velocity potential in the physical plane

$$\varphi_{xx}\left(1 - \frac{\varphi_x^2}{a^2}\right) + \varphi_{yy}\left(1 - \frac{\varphi_y^2}{a^2}\right) - 2\varphi_{xy}\,\frac{\varphi_x\varphi_y}{a^2} = 0$$

is transformed into the hodograph equation

$$\chi_{vv}\left(1 - \frac{u^2}{a^2}\right) + \chi_{uu}\left(1 - \frac{v^2}{a^2}\right) + 2\chi_{uv}\,\frac{uv}{a^2} = 0$$

V

Equations of motion

1. Euler's equations (w = component of velocity in z direction)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Where X, Y, and Z are the components of extraneous forces per unit mass at point x, y, z at time t.

2. Lagrangian forms of the dynamical equations

By use of the Lagrangian equations

$$\frac{\partial^2 x}{\partial t^2} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial^2 y}{\partial t^2} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial^2 z}{\partial t^2} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Where a, b, and c are initial coordinates of any particle of fluid (hence a, b, c, and t are the independent variables). There are obtained the following forms of Euler's equations:

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial a} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial a} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial p}{\partial a} = 0$$

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial b} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial b} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial b} + \frac{1}{\rho}\frac{\partial p}{\partial b} = 0$$

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial c} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial c} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial c} + \frac{1}{\rho}\frac{\partial p}{\partial c} = 0$$

3. Equationsof motion for two dimensional steady motion in polar coordinates

$$\text{(a)} \quad \begin{cases} u\dfrac{\partial u}{\partial r} + \dfrac{v}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} \\[2ex] u\dfrac{\partial v}{\partial r} + \dfrac{v}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{uv}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{r\partial \theta} \end{cases}$$

By means of the relations $v = \omega r$, $\frac{\partial r}{\partial t} = u$, and

$$\alpha = \frac{\partial \omega}{\partial t}$$

the equations 3(a) become

$$\text{(b)} \quad \frac{\partial u}{\partial t} - r\omega^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$2u\omega + r\alpha = -\frac{1}{\rho}\frac{\partial p}{r\partial \theta}$$

Where $\frac{\partial u}{\partial t}$ is the time derivative of the velocity as the particle moves along a streamline and $\frac{\partial u}{\partial t}$ is zero at any fixed point in the flow (steady motion condition).

4. Equation of motion in one dimension with no extraneous forces $(X = Y = Z = 0)$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial x} = 0$$

in which p is a unique function of ρ

5. The Bernoulli equation

A. Differential forms

(a) Simplification of Euler's equations by considering one-dimensional steady motion with $X = Y = Z = 0$ yields

$$\frac{dp}{\rho} + w\,dw = 0$$

where flow in a stream tube is considered and w is the magnitude of the resultant velocity as defined under notation.

(b) Differential equation expressed in terms of velocity of sound

$$a^2\frac{d\rho}{\rho} + w\,dw = 0$$

or $$\frac{2}{\gamma-1}\,a\,da + w\,dw = 0$$

B. Integrated forms of the Bernoulli equation

(c) General Bernoulli equation for compressible flow

$$\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{{w_1}^2}{2} = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{{w_2}^2}{2} = \text{constant}$$

or $$\frac{p_1}{p_2} = \left[1 - \frac{\gamma-1}{2}\left(\frac{w_1^2 - w_2^2}{a_2^2}\right)\right]^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}M_1^2\left(1 - \frac{w_2^2}{w_1^2}\right)\right]^{\frac{\gamma}{1-\gamma}}$$

(d) Critical pressure ratio or pressure ratio corresponding to attainment of sonic velocity at some point in the field of flow (equation 5(c) expressed in terms of local and stream conditions for a local Mach number of unity)

$$\frac{p_l}{p_s} = \left(1 + \frac{\gamma-1}{2}M_s^2\right)^{\frac{\gamma}{\gamma-1}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

(e) Equations relating pressures at stagnation point and any other point along a streamline

$$\frac{p_o}{p} = \left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}$$

or

$$\frac{p_o}{p} = \left[1 - \frac{\gamma-1}{2}\frac{w^2}{a_o^2}\right]^{\frac{\gamma}{1-\gamma}}$$

(f) The compressibility factor

If
$$\frac{\gamma-1}{2}M^2 < 1 \left(\text{or } M < \sqrt{\frac{2}{\gamma-1}}\right),$$
5(e) may be expanded in a binomial series

$$p_o = p + \frac{1}{2}\rho w^2\left[1 + \frac{1}{2!2}M^2 + \frac{1(2-\gamma)}{3!2^2}M^4 + \frac{1(2-\gamma)(3-2\gamma)}{4!2^3}M^6 + \ldots\right.$$

and with $\gamma = 1.4$, the compressibility factor

$$F = \frac{p_o - p}{\frac{1}{2}\rho w^2}$$

becomes

$$1 + \frac{1}{4}M^2 + \frac{1}{40}M^4 + \ldots$$

(g) Expressions for the velocity at any point along a streamline

Upon integration of 5(b) there is obtained

$$w^2 = \frac{2}{\gamma-1}\left(a_o^2 - a^2\right)$$

or

$$w = a_o\sqrt{\frac{2}{\gamma-1}}\sqrt{1 - \left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}}}$$

(h) The maximum attainable velocity is given by (expansion to zero pressure)

$$w_m^2 = \frac{2\gamma}{\gamma-1}\frac{p}{\rho} + w^2 = \frac{2a^2}{\gamma-1} + w^2$$

(j) The Bernoulli equation in pressure coefficient form

$$\frac{p_l - p_s}{\frac{1}{2}\rho_s W_s^2} \doteqdot P = \frac{2}{\gamma M_s^2}\left[1 - \frac{\gamma-1}{2}\left(\frac{w_l^2}{w_s^2} - 1\right) M_s^2\right]^{\frac{\gamma}{\gamma-1}} - \frac{2}{\gamma M_s^2}$$

(k) Expression for velocity ratio in terms of pressure coefficient and stream Mach number

$$\frac{w_l}{w_s} = \sqrt{1 - \frac{2}{(\gamma-1)M_s^2}\left[\left(1 + \frac{\gamma}{2}PM_s^2\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

and
$$\left(\frac{w_{cr}}{w_s}\right)^2 = \frac{2}{\gamma+1}\frac{1}{M_s^2} + \frac{\gamma-1}{\gamma+1}$$

where $w_{cr} = a_l$ (or $M_l = 1$)

(l) Expression for local Mach number in terms of stream Mach number and pressure coefficient

$$M_l = \sqrt{\frac{M_s^2 - \frac{2}{\gamma-1}\left(\beta^{\frac{\gamma-1}{\gamma}} - 1\right)}{\beta^{\frac{\gamma-1}{\gamma}}}}$$

where $\beta = 1 + \frac{\gamma}{2} PM_s^2$

(m) Expression for acceleration along a streamline (one-dimensional steady motion)

incompressible flow

$$\frac{dw}{dt} = - \frac{w_s^2}{2}\frac{dP}{dx}$$

compressible flow

$$\frac{dw}{dt} = - \frac{w_s^2}{2}\left(1 + \frac{\gamma}{2}PM_s^2\right)^{-\frac{1}{\gamma}}\frac{dP}{dx}$$

(n) General expression for the acceleration of a fluid particle (two-dimensional steady motion)

$$\frac{ru}{a^2}\left(\frac{\partial u}{\partial t} - r\omega^2\right) - \frac{\partial u}{\partial r}r - u + \frac{r^2\omega}{a^2}\left(r\alpha + 2\omega u\right) - r\frac{\partial \omega}{\partial \theta} = 0$$

The above expression is obtained by combining the expressions for radial and transverse accelerations (equations 3(b) page 13) with the equation of continuity (equation 3 page 19).

(c) Explicit form of the general expression for the acceleration of a fluid particle (two-dimensional steady motion).

If a reference system is chosen so that the radial component of acceleration is zero, the general expression for acceleration may be written

$$\text{Acceleration} = \frac{w^2}{r\omega} \frac{\frac{\partial u}{\partial r}}{\left(\frac{r\omega}{a}\right)^2 - 1}$$

6. Conditions for irrotational motion

(a) Equation in rectangular coordinates (two dimensions)

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

or $$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad (\nabla^2\psi = 0)$$

(b) Equation in polar coordinates

$$2\omega + r\frac{\partial \omega}{\partial r} - \frac{\partial u}{r\partial \theta} = 0$$

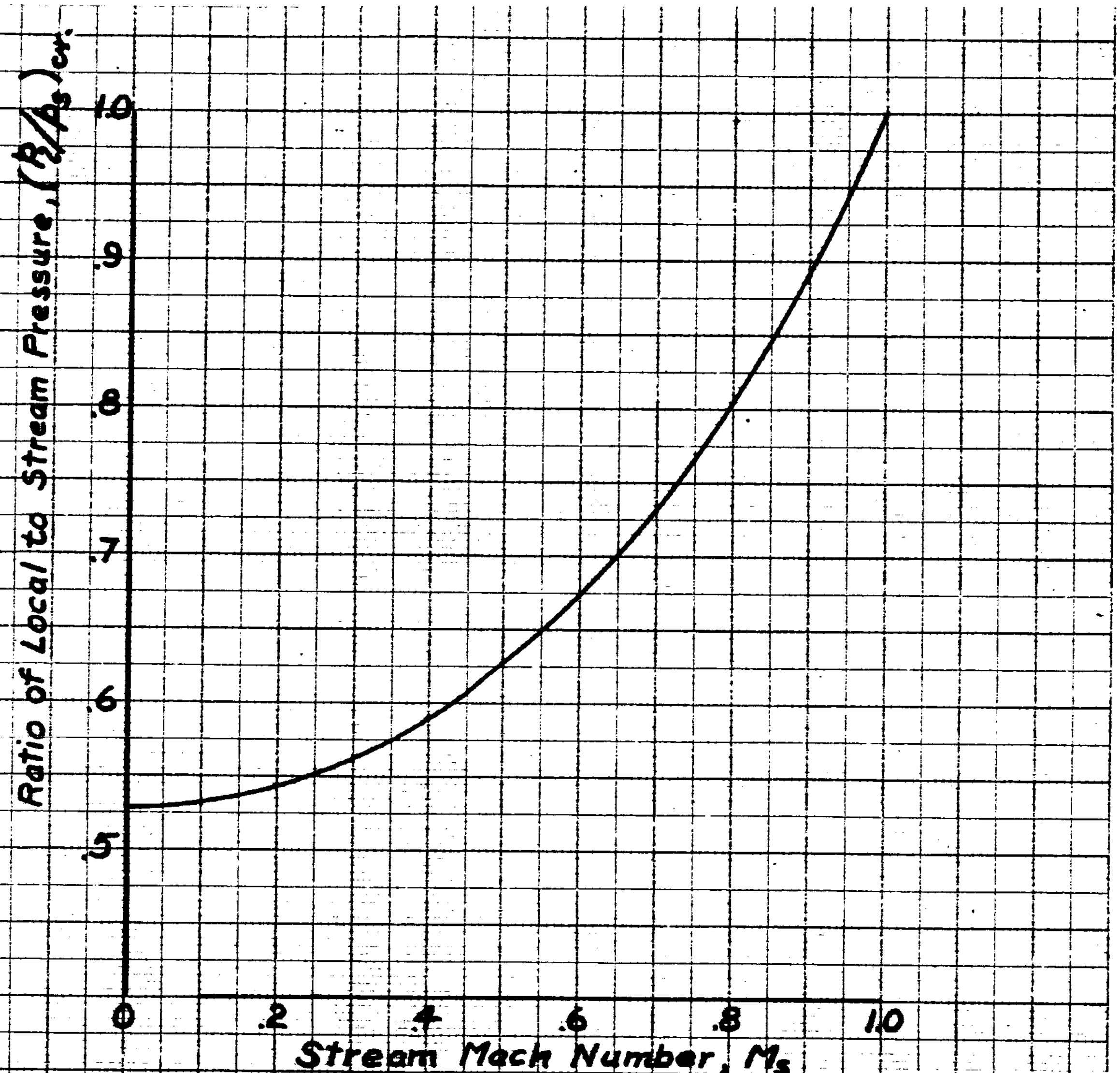


Figure V a — Critical pressure ratio as a function of stream mach number.

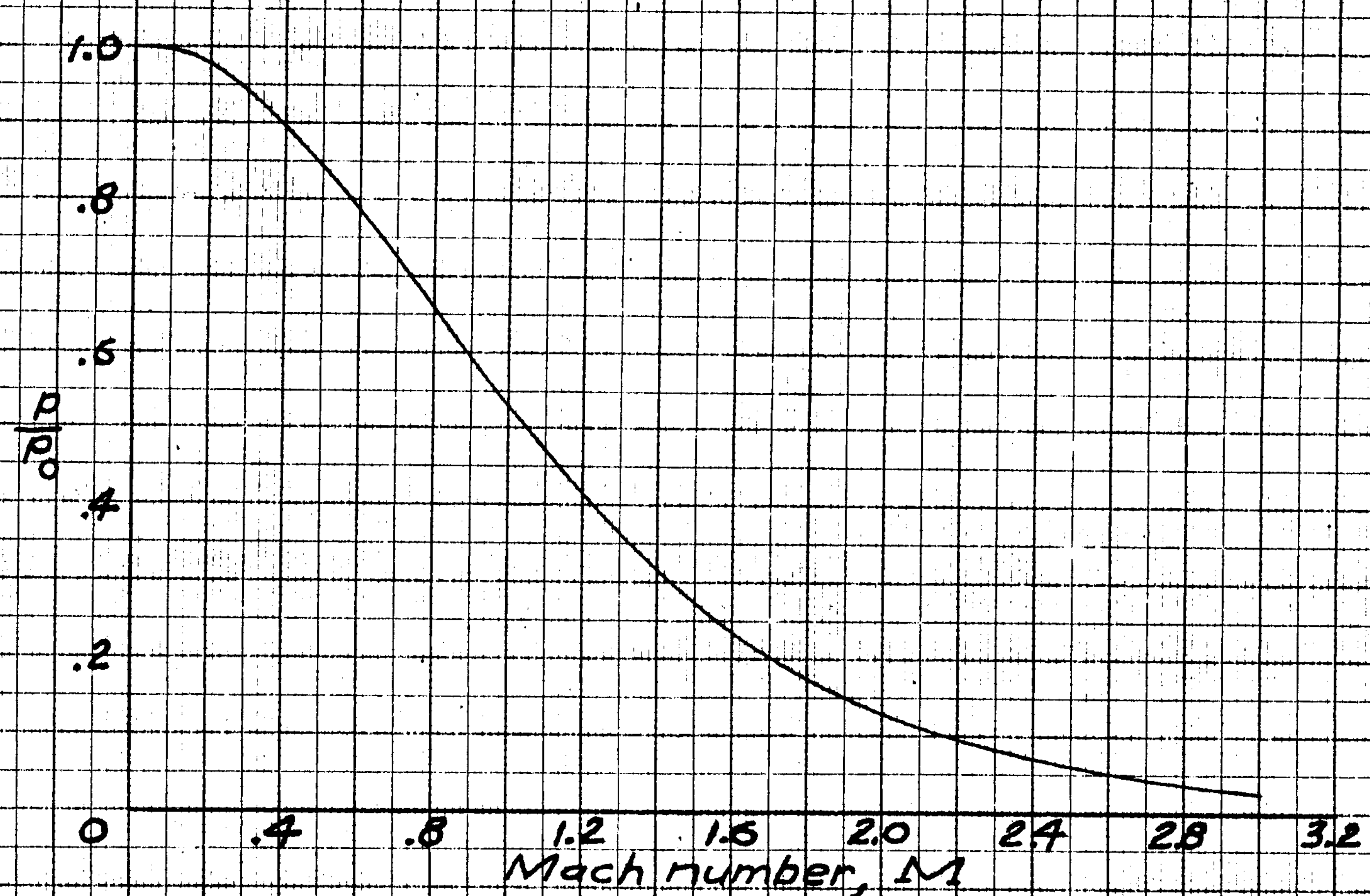


Figure VI.- Variation of pressure ratio with Mach number.

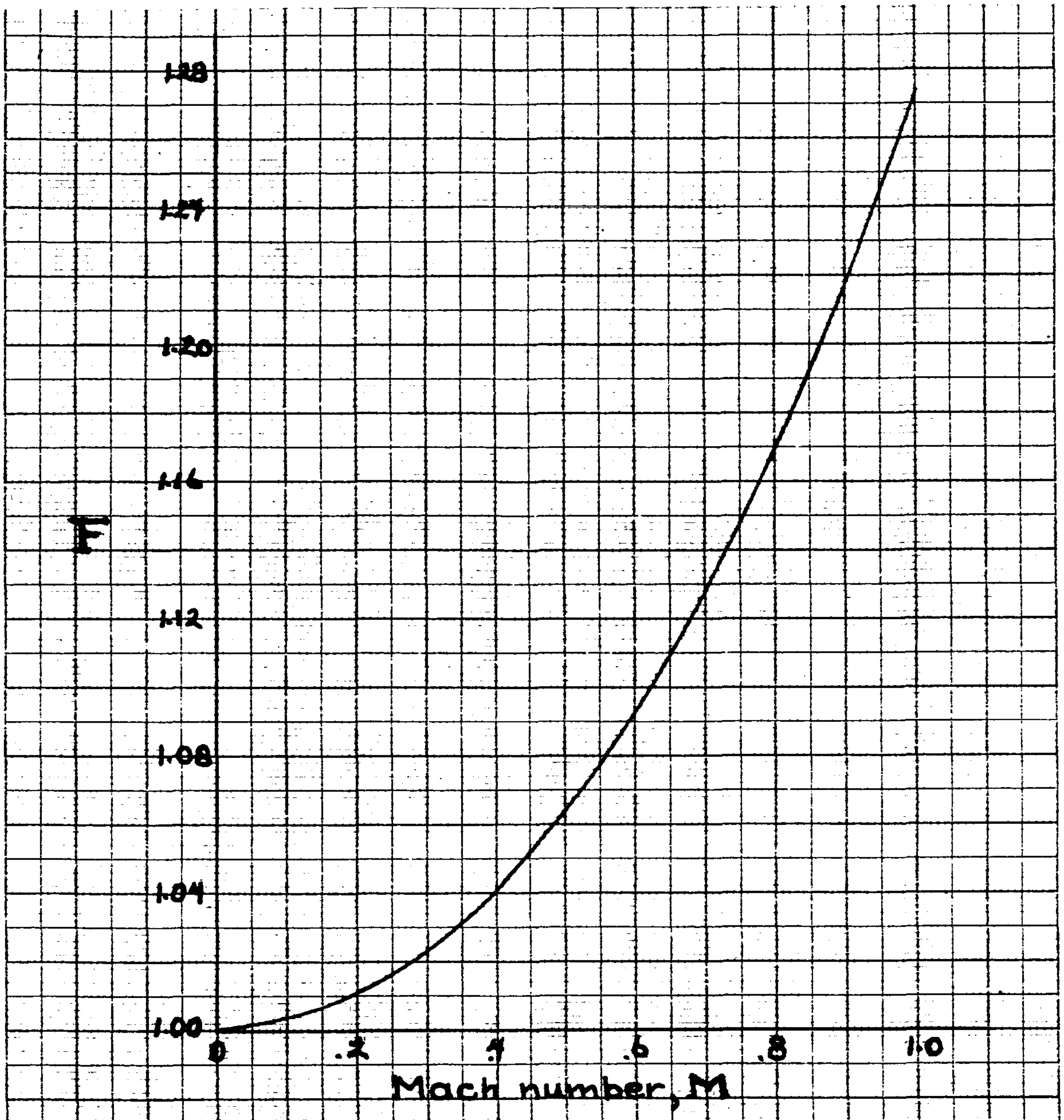


Figure V-c.—Compressibility factor

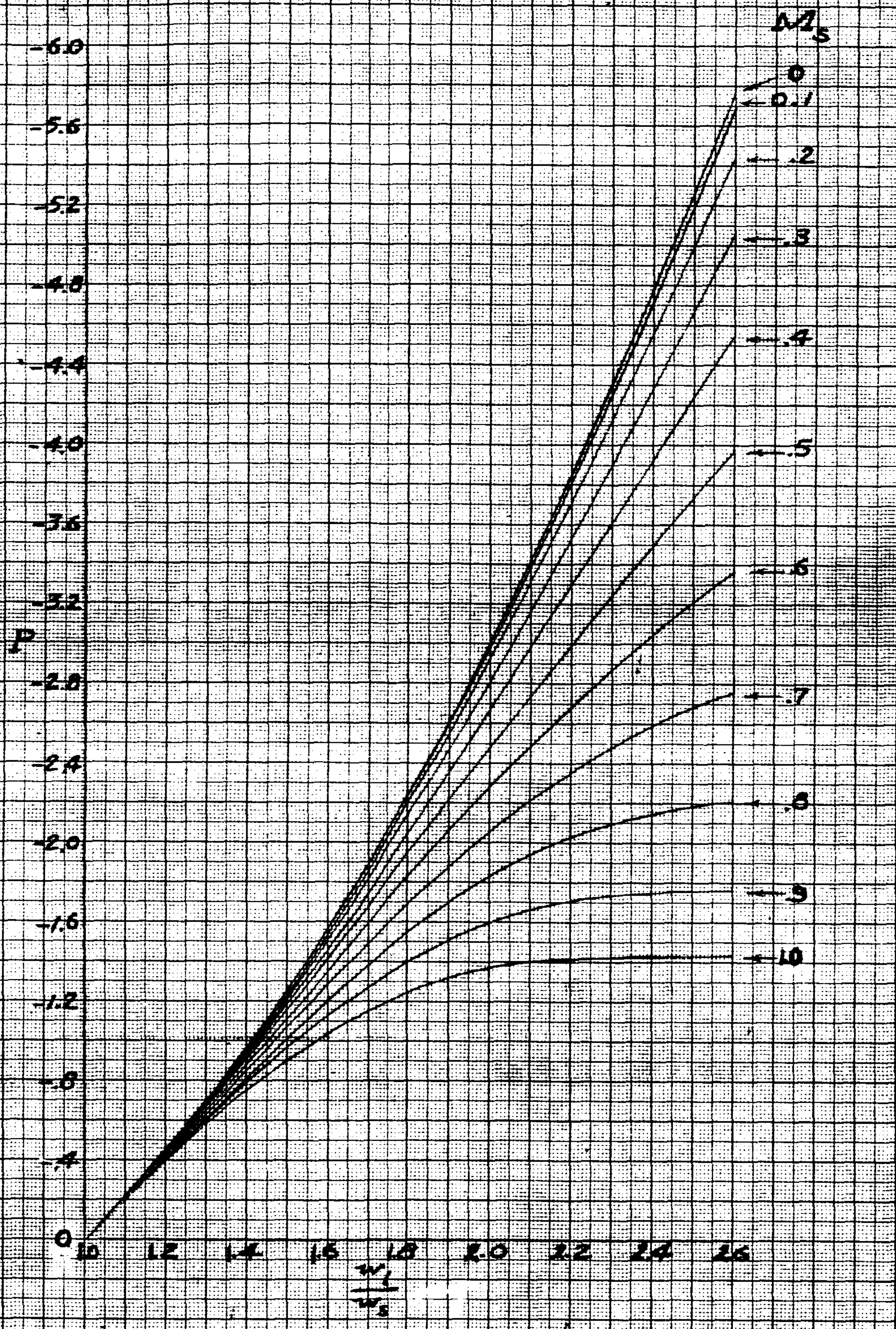


Figure VI.- Pressures corresponding to given velocity ratios.

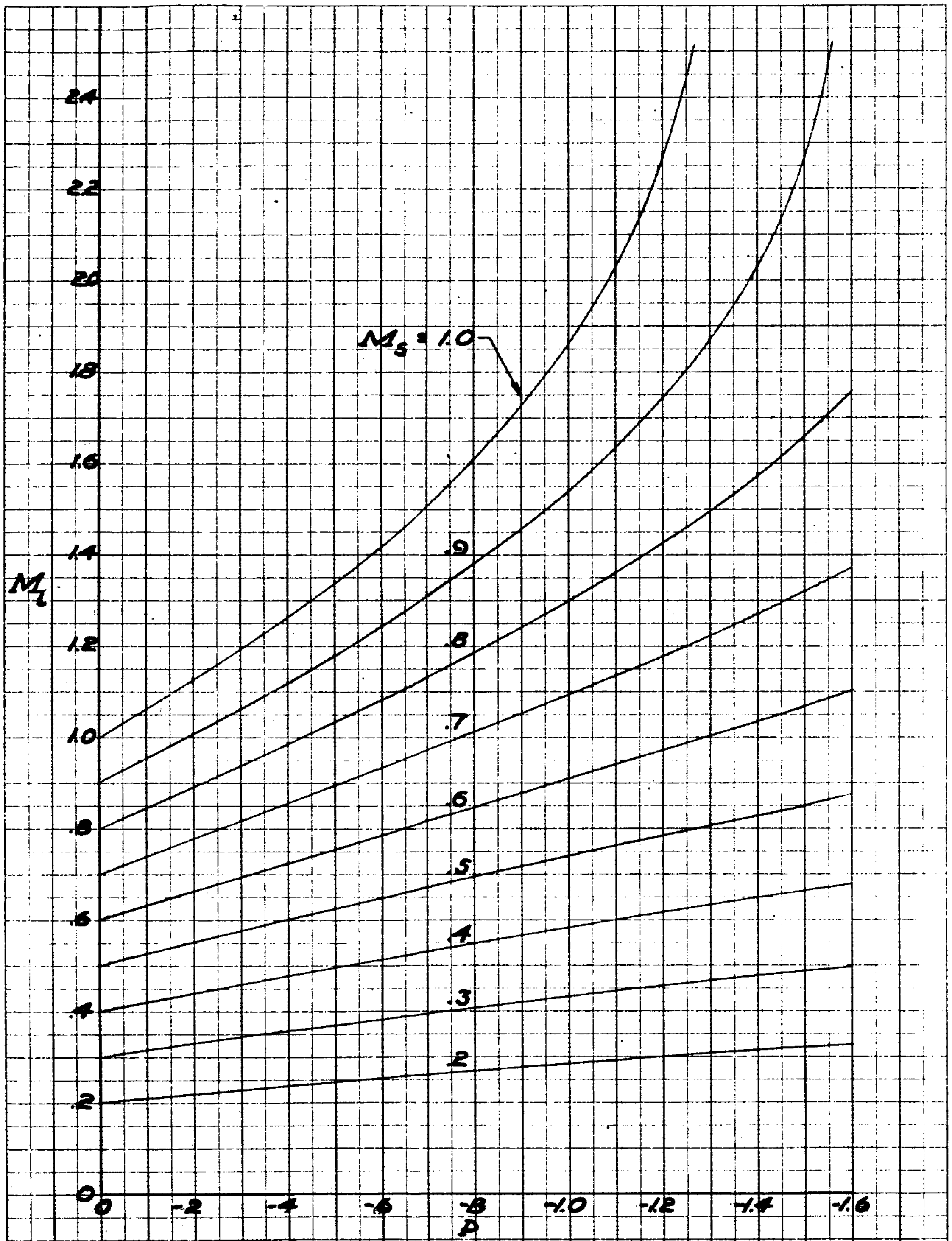


Figure V e.- Local Mach number as a function of stream Mach number and pressure coefficient.

VI

Equations of continuity

1. Euler's form

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

(a) For steady incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(b) For one-dimensional motion

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

2. Lagrangian form (equation of continuity at time, t)

$$\rho\frac{\partial(x,y,z)}{\partial(a,b,c)} = \rho_o$$

Where ρ_o = initial density at point a,b,c (as defined under equations of motion).

(a) For an incompressible fluid

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = 1$$

3. Polar form for two-dimensional steady motion

$$\frac{\partial(\rho ur)}{\partial r} + \frac{\partial(\rho r\omega)}{\partial\theta} = 0; \text{or } \frac{\partial(\rho ur)}{\partial r} + \frac{\partial(\rho v)}{\partial\theta} = 0$$

4. The continuity conditions within a stream tube are defined by:

ρwS = constant (steady-motion condition)

5. In two-dimensional steady motion, the continuity conditions at a fixed point in the flow plane are given by

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

6. By differentation of 4 there is obtained:

$$\frac{dS}{S} + \frac{dw}{w} + \frac{d\rho}{\rho} = 0$$

(a) At minimum section $dS = 0$

hence: $\frac{dw}{w} = - \frac{d\rho}{\rho}$

(b) Substituting in Bernoulli's equation there is obtained the condition for maximum mass flow:

$$w_t = \sqrt{\frac{dp}{d\rho}} = a_t$$

Hence the velocity at the minimum section is equal to the local velocity of sound.

VII

Combination of the Equations of Motion and Continuity

1. Differential equations

$$\frac{dS}{S} = - \frac{dw}{w}(1-M^2) = \frac{1-M^2}{\gamma M^2}\frac{dp}{p}$$

2. Stream tube area ratios

$$\frac{S_s}{S_l} = \frac{\left(\alpha\beta^{\frac{2}{\gamma}} - \beta^{\frac{1+\gamma}{\gamma}}\right)^{\frac{1}{2}}}{\left(\frac{\gamma-1}{2}\right)^{\frac{1}{2}} M_s}$$

where

$$\alpha = 1 + \frac{\gamma-1}{2} M_s{}^2$$

and

$$\beta = 1 + \frac{\gamma}{2} PM_s{}^2 = \frac{p_l}{p_s}$$

3. Mass flow per unit time through a stream tube or convergent-divergent nozzle

$$m = S\sqrt{2\frac{\gamma}{\gamma-1}\frac{p_o}{V_o}\left[\left(\frac{p}{p_o}\right)^{\frac{2}{\gamma}} - \left(\frac{p}{p_o}\right)^{\frac{\gamma+1}{\gamma}}\right]}$$

4. Pressure ratio for maximum mass flow

By differentiation of (3) there is obtained the condition for maximum mass flow

$$\frac{p}{p_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

5. The maximum mass flow is

$$m = S_t \sqrt{\gamma \frac{p_o}{V_o} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

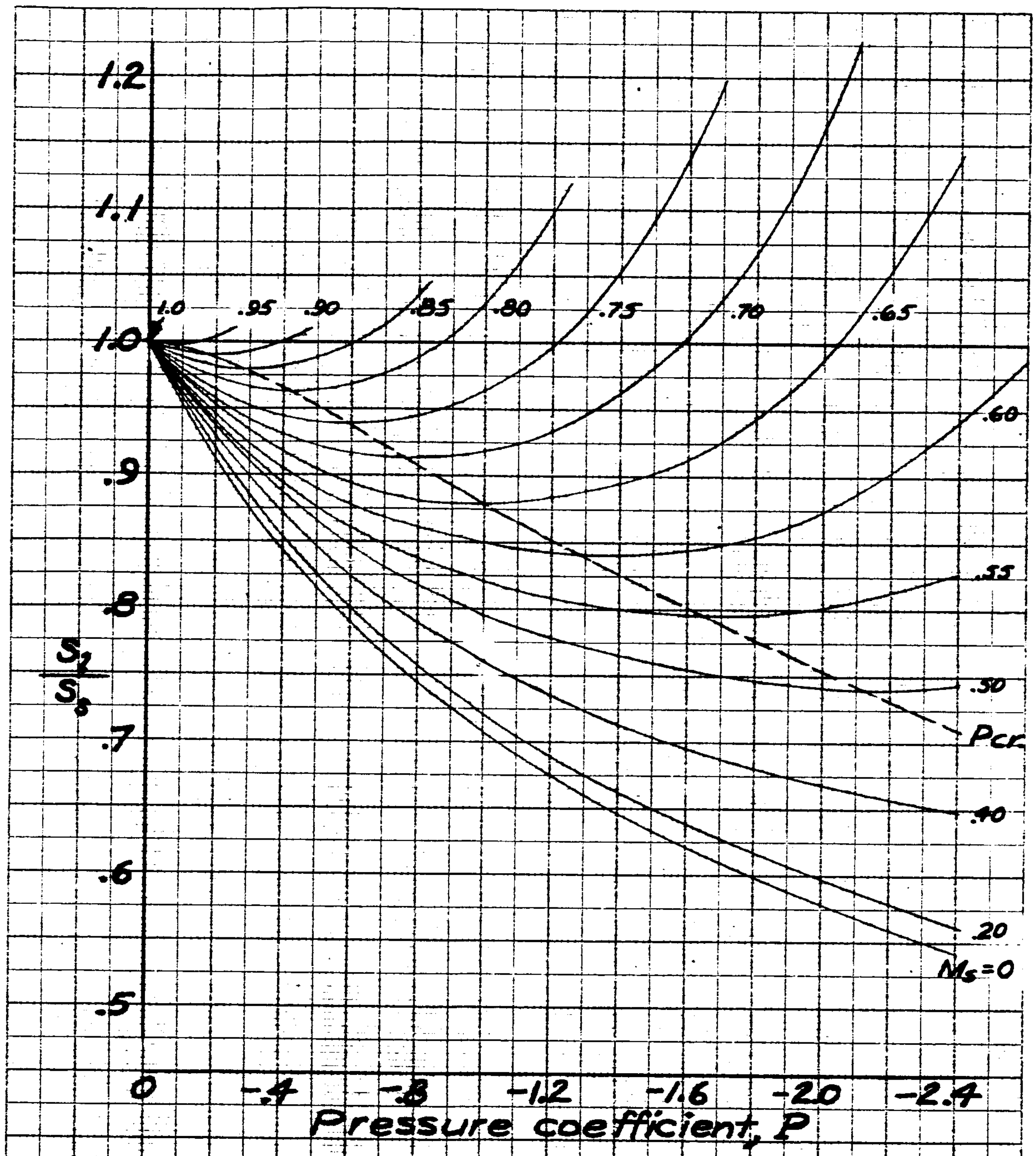


Figure VIIIa.— Variation of streamtube area ratio with pressure coefficient.

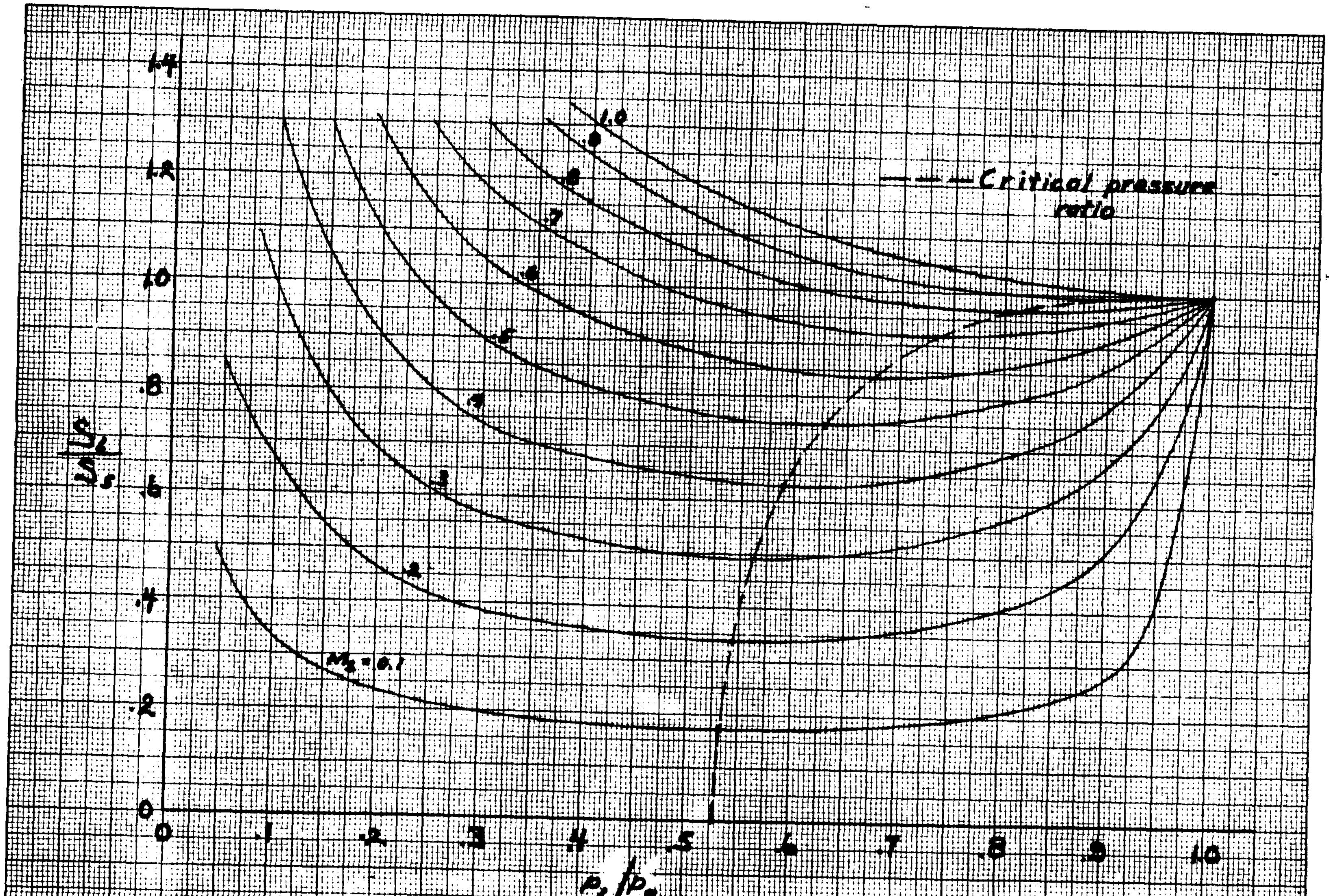


Figure VII-6—Variation of streamtube area ratio with pressure ratio.

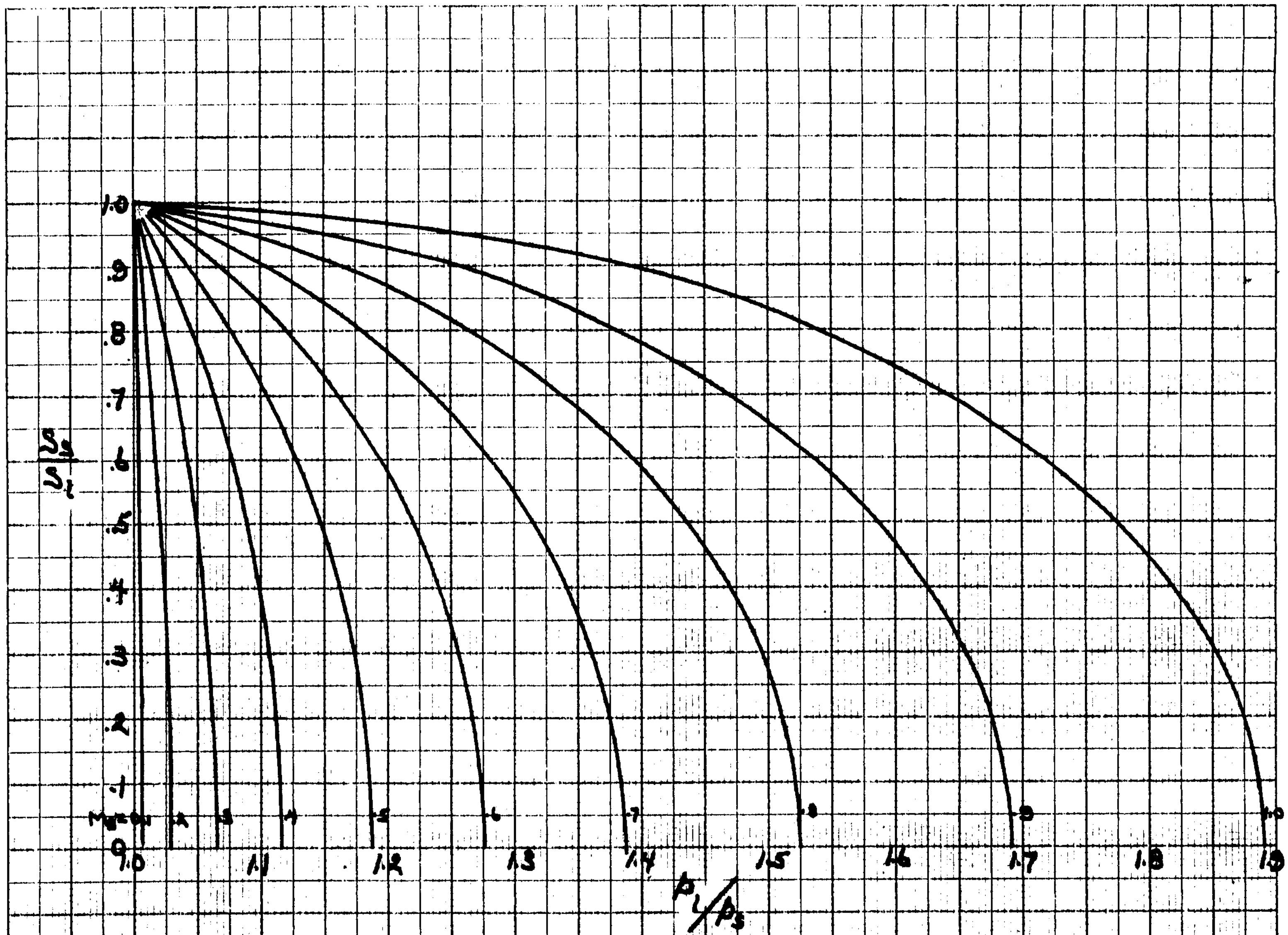


Figure VIIa; — Variation of stream tube area ratio with pressure ratio.

VIII

Pressure Coefficient (expressions for)

1. Definition

$$P = \frac{p_l - p_s}{\frac{1}{2}\rho_s w_s^2} \left(= \frac{\Delta p}{q} \right)$$

2. Pressure coefficient in terms of velocity ratios

(a) Incompressible flow

$$P = 1 - \left(\frac{w_l}{w_s} \right)^2$$

(b) Compressible flow

$$P = \frac{2}{\gamma M_s^2} \left[1 - \frac{\gamma - 1}{2} \left(\frac{w_l^2}{w_s^2} - 1 \right) M_s^2 \right]^{\frac{\gamma}{\gamma - 1}} - \frac{2}{\gamma M_s^2}$$

$$\text{or} \quad P = \left[1 - \left(\frac{w_l}{w_s} \right)^2 \right] + \frac{M_s^2}{4} \left[1 - \left(\frac{w_l}{w_s} \right)^2 \right]^2 + \ldots$$

3. Pressure coefficient in terms of pressure ratios

(a)
$$P = F_s \left[\frac{\frac{p_l}{p_o} - \frac{p_s}{p_o}}{1 - \frac{p_s}{p_o}} \right]$$

(b) In terms of local and stream pressures

$$P = \frac{2}{\gamma M_s^2} \left[\frac{p_l}{p_s} - 1 \right]$$

(c) In terms of local and stagnation pressures

$$P = \frac{2}{\gamma M_s^2}\left[\frac{p_l}{p_o}\left(1 + \frac{\gamma-1}{2} M_s^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

4. The maximum negative pressure coefficient

   (pressure coefficient for $p_l = 0$)

$$P_m = -\frac{2}{\gamma M_s^2}$$

5. Critical pressure coefficient or pressure coefficient corresponding to attainment of local speed of sound at some point in the field of flow

$$P_{cr} = \frac{2}{\gamma M_s^2}\left[\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}\left(1 + \frac{\gamma-1}{2} M_s^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

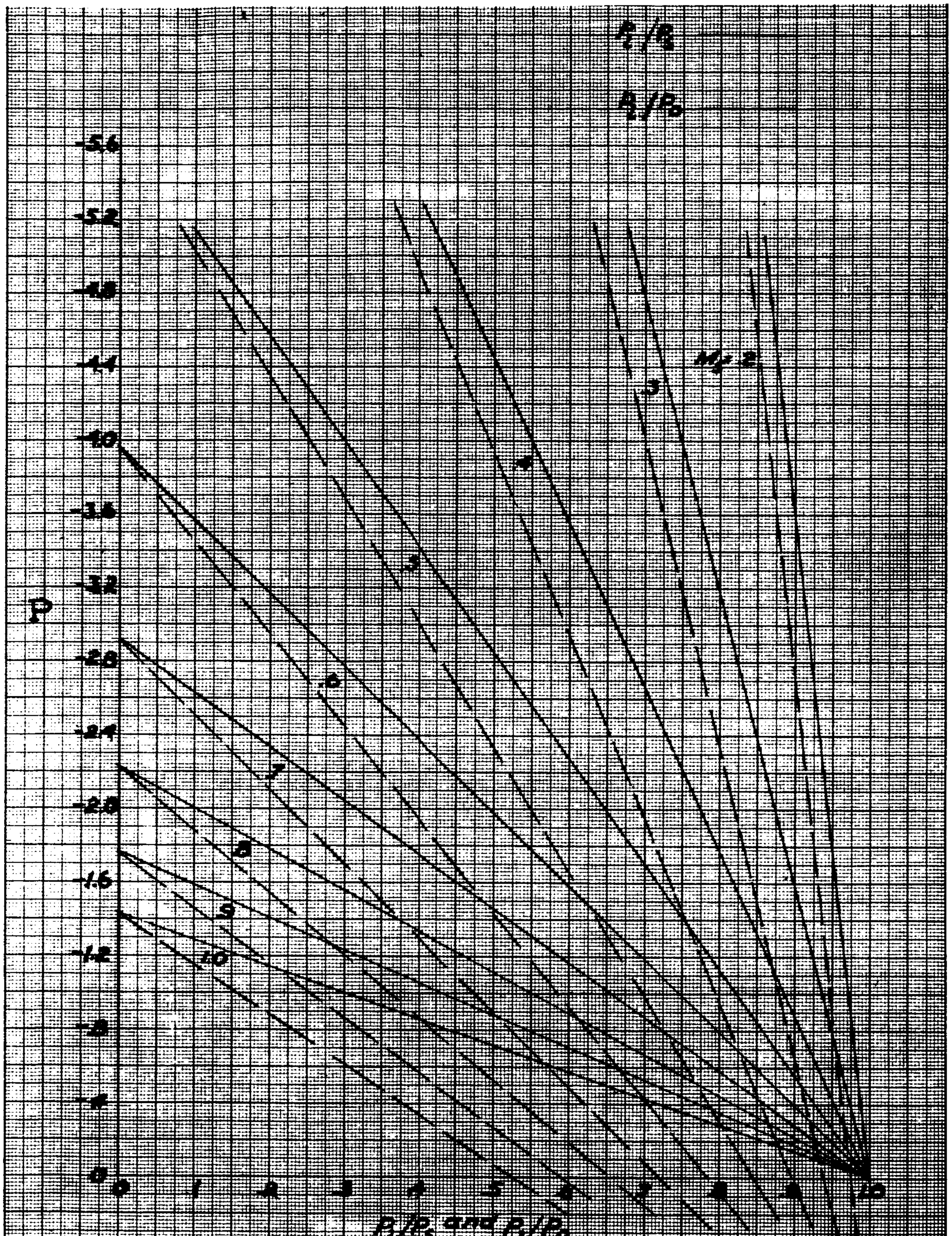


Figure XIII a—Pressure coefficient as a function of pressure ratio and stream Mach number.

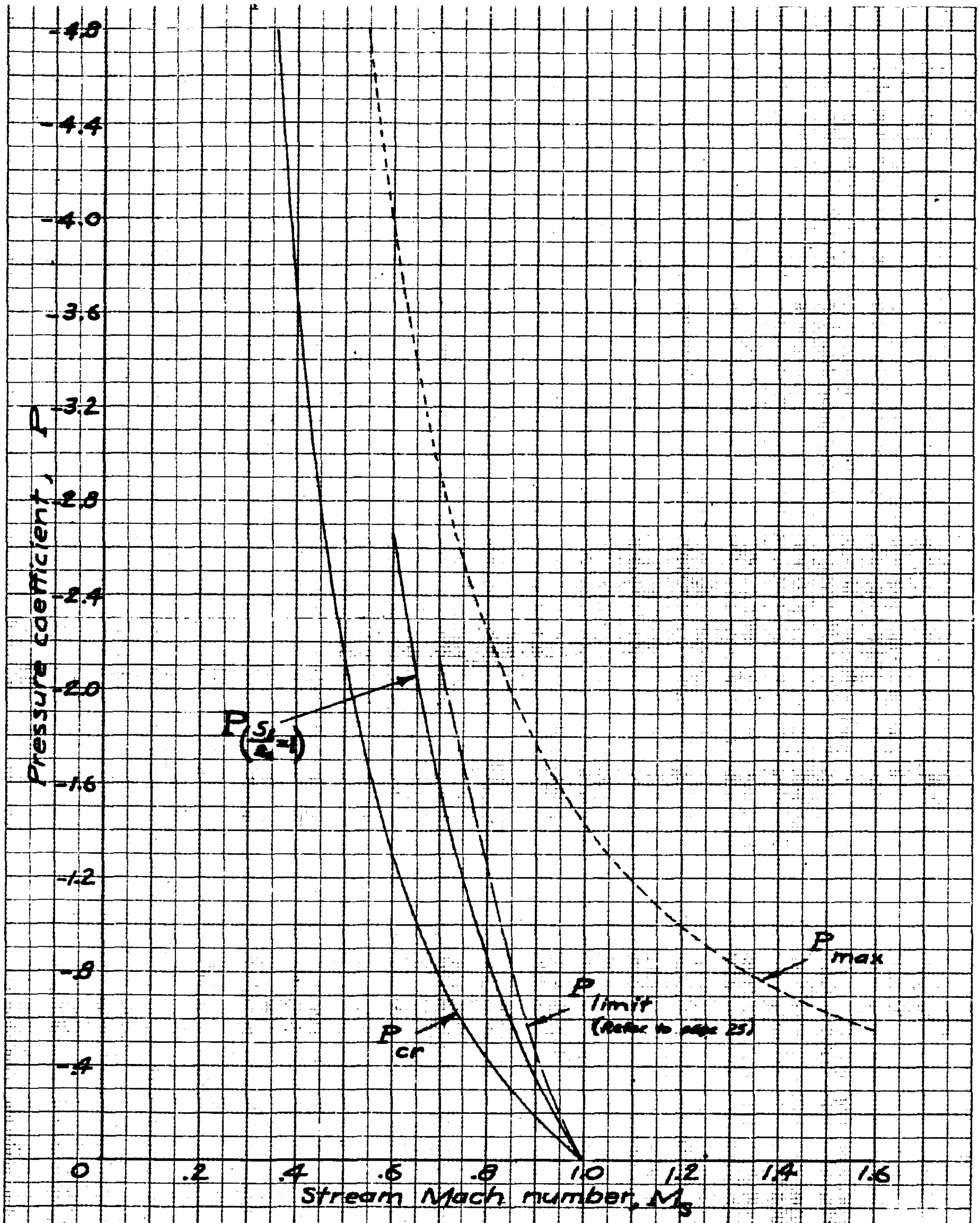


Figure VIII. – Variation of pressure coefficient with stream Mach number for several flow conditions.

IX

Subsonic Two-Dimensional Compressible Flow

A. Solutions in the physical plane

1. Equations for the velocity potential

(incompressible flow $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$)

For two-dimensional compressible flow

$$\text{(a)}\quad \varphi_{xx}\left(1 - \frac{\varphi_x^2}{a_l^2}\right) + \varphi_{yy}\left(1 - \frac{\varphi_y^2}{a_l^2}\right) - \frac{2\varphi_x\,\varphi_y}{a_l^2}\,\varphi_{xy} = 0$$

or

$$\text{(b)}\quad \left(\frac{a_l^2}{a_s^2} - M_s^2\,\bar{u}^2\right)\bar{\varphi}_{xx} + \left(\frac{a_l^2}{a_s^2} - M_s^2\,\bar{v}^2\right)\bar{\varphi}_{yy} - 2M_s^2\,\bar{\varphi}_x\,\bar{\varphi}_y\,\bar{\varphi}_{xy} = 0$$

where

$$\bar{\varphi}_x = \frac{\varphi_x}{w_s},\quad \bar{\varphi}_{\substack{xx\\yy}} = \frac{\varphi_{\substack{xx\\yy}}}{w_s} \quad\text{and}\quad \bar{u} = \frac{u}{w_s} \quad \text{etc.}$$

or

$$\text{(c)}\quad \left(\varphi_{xx} + \varphi_{yy}\right)\left[2a_s^2 - (\gamma-1)\left(w_l^2 - w_s^2\right)\right]$$
$$= \varphi_x \frac{\partial w_l^2}{\partial x} + \varphi_y \frac{\partial w_l^2}{\partial y}$$

No general solution has been obtained for subsonic flow, but these equations have been solved completely for special problems in supersonic flow.

2. Prandtl obtains an approximation for thin bodies at low Mach numbers by assuming

$$v_l \ll u_l < a_l \quad \text{or} \quad u_l \cong u_s \quad \text{and} \quad a_l \cong a_s$$

(a) Imposing these conditions on equation 1(a) there is obtained

$$\varphi_{xx}\left(1-M_s^2\right) + \varphi_{yy} = 0$$

(b) For incompressible flow in the same coordinates

$$\varphi_{xx} + \varphi_{yy} = 0$$

(c) Applying the transformation $\xi = x$ and $\eta = y\sqrt{1-M^2}$ to 2(a) there is obtained $\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0$ which is 2(b), the incompressible case. It is found that the effect of compressibility is to expand the field of flow in the ratio $\frac{1}{\sqrt{1-M^2}}$ and to increase the induced velocity ratios by the same factor. Since the body is assumed thin and the induced velocities therefore small, the pressure coefficient, P, also will vary with Mach number according to $\frac{1}{\sqrt{1-M^2}}$

3. Equation 1(c) in polar coordinates

(a) $$\nabla^2\varphi\left[2a_s^2 - (\gamma-1)(w_l^2 - w_s^2)\right] = \frac{\partial\varphi}{\partial r}\frac{\partial w_l^2}{\partial r} + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\frac{\partial w_l^2}{\partial\theta}$$

4. Method of Rayleigh and Janzen

A first approximation to actual flow conditions is obtained from the incompressible case $\nabla^2\varphi = 0$. Using particular solutions of $\nabla^2\phi = 0$ of type $\phi = r^n \cos n\theta$, the right hand side of 3(a) can be calculated as a function of r and $\theta = F$; this method is repeated for higher approximations.

Second approximation

$$\nabla^2\Phi = \frac{1}{2a_s^2} F(r,\theta) \text{ etc.}$$

5. Method of Poggi (NACA T. R. No. 624)

Poggi considers compressible flow to be replaced by an incompressible flow due to a distribution of sinks and sources throughout the region of flow.

Strength of distribution in the plane of the profile

$$-\frac{1}{4\pi a_l^2}\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial w^2}{\partial\xi} + \frac{\partial\varphi}{\partial\eta}\frac{\partial w^2}{\partial\eta}\right)d\xi\ d\eta$$

and in the plane of the circle into which the profile is mapped

$$\frac{1}{4\pi a_l^2}\left(u\frac{\partial w^2}{\partial\lambda} - \frac{v}{\lambda}\frac{\partial w^2}{\partial\theta}\right)\frac{R}{\lambda}\ d\lambda\ d\theta$$

where r and θ are the polar coordinates in the plane of the circle, R and δ are the radius of the circle into which profile is mapped and the angular coordinate on this circle respectively. $\left(\lambda = \frac{R}{r},\ v = -\frac{\partial\varphi}{\partial r} \text{ and } u = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$, and w is the magnitude of the velocity in the plane of the profile.

The total induced velocity at any point $P(R, \delta)$ of the circular boundary by foregoing system of sources and sinks is

$$\Delta w = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{u \frac{\partial w^2}{\partial \lambda} - \frac{v}{\lambda} \frac{\partial w^2}{\partial \theta}}{a_l^2 \left(1-2\lambda \cos (\theta-\delta) + \lambda^2\right)} \sin (\theta-\delta)\, d\lambda\, d\theta$$

As the first approximation Δw is obtained from the substitution in above expression of incompressible values of v, u, and w^2.

then

$$w_{comp.} = w_{incomp.} + \Delta w$$

With the simplification $a_{local} = a_{stream}$, the expression for Δw becomes

$$\Delta w = \frac{1}{2\pi a_s^2} \int_0^1 \int_0^{2\pi} \frac{u \frac{\partial w^2}{\partial \lambda} - \frac{v}{\lambda} \frac{\partial w^2}{\partial \theta}}{1 - 2\lambda \cos (\theta-\delta) + \lambda^2} \sin (\theta-\delta)\, d\lambda\, d\theta$$

6. Method of Kaplan

In a recent paper, Kaplan has succeeded in obtaining a solution in the form of a series in t(thickness ratio of body) which is accurate for fairly thick shapes up to stream Mach numbers of unity; he obtains a solution exact in M_s whereas Poggi gets a solution in the form of a series in M_s which is exact in t(thickness ratio of body). Poggi's method is slowly convergent in M_s, and hence, since we are almost always interested in high-speed flow about bodies for which $t < 0.3$, Kaplan's method is superior. This method establishes a limiting value of the negative pressure coefficient for each Mach number (see fig. VIII b).

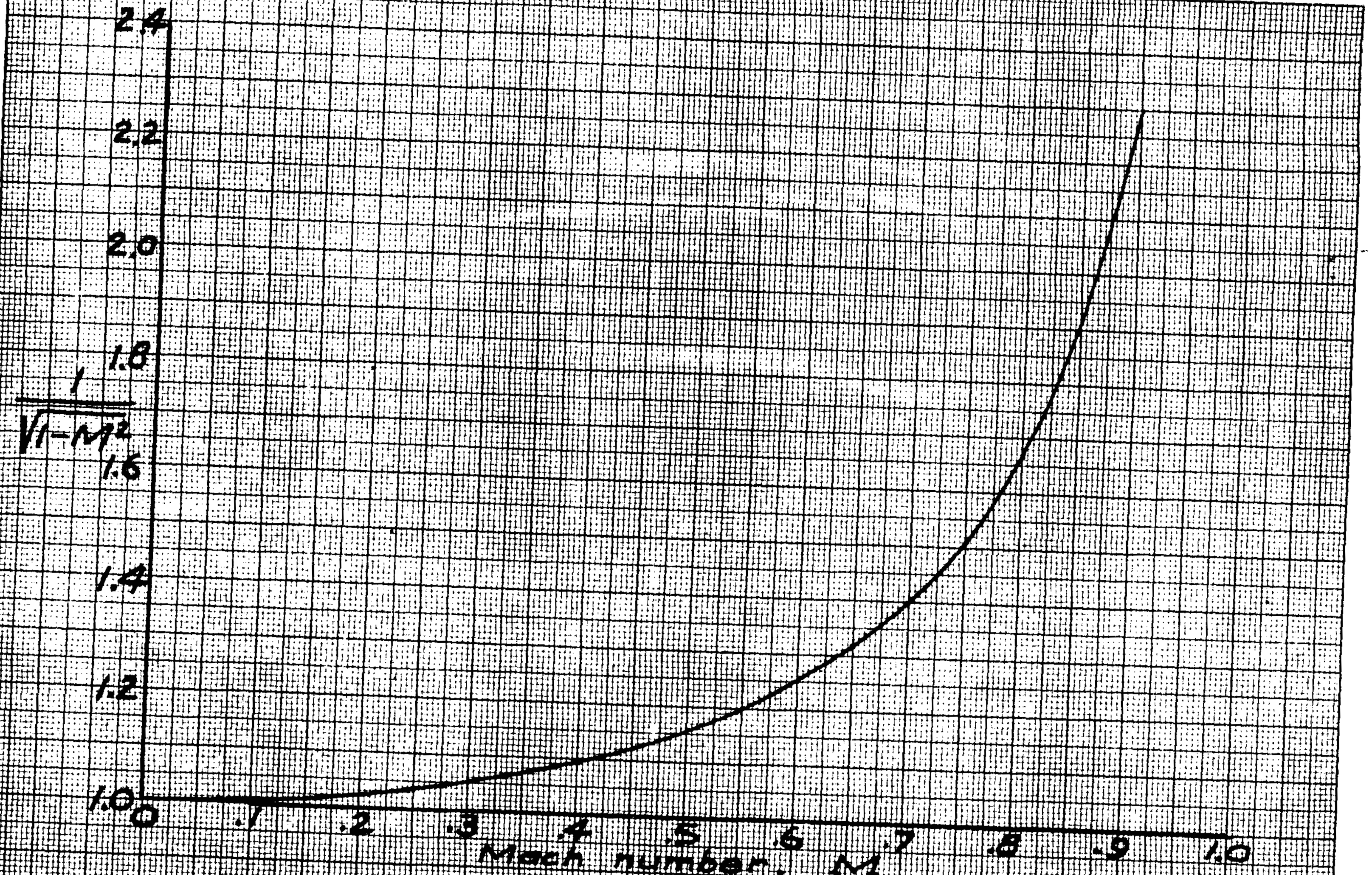


Figure IXa — The Glauert-Prandtl Factor

Solutions in the Velocity Plane

The hodograph method

1. Introduction (as given by Temple and Yarwood)

The hodograph method due to Molenbroek and Chaplygin is a method of constructing steady irrotational two-dimensional fields of flow of compressible fluids in which the pressure is a prescribed function of the density. In particular it is applicable to adiabatic or isothermal flow, and, in the case of adiabatic flow, it is applicable to both subsonic and supersonic conditions. The great advantage of the hodograph method, is that given a steady irrotational field of flow of an incompressible fluid around prescribed boundaries, it determines exactly a field of compressible flow around boundaries of approximately the same shape. Thus, given the incompressible flow around an airfoil, the hodograph method determines the compressible flow around an airfoil of approximately the same form.

The hodograph method suffers, however, from two disadvantages: In the first place, the boundaries in the field of incompressible and compressible flow are not exactly the same. The boundaries in the compressible flow are slightly distorted by an amount increasing with the Mach number in the main stream. In the second place, the field of compressible flow is not given in a form suitable for numerical computation. The expressions obtained for the velocity potential and stream function are exact but they are in the form of an infinite series which cannot be easily summed analytically or numerically.

2. General hodograph equations

(a) Equation obtained from expression for φ in the physical plane by means of the Legendre contact transformation for two independent variables

$$\chi_{vv}\left(1 - \frac{u^2}{a^2}\right) + \chi_{uu}\left(1 - \frac{v^2}{a^2}\right) + 2\chi_{uv}\frac{uv}{a^2} = 0$$

where

$$\chi = x\varphi_x + y\varphi_y - \varphi$$

(b) General hodograph equations for a two-dimensional compressible gas.

From the defining expressions

$$d\varphi + i\left(\frac{\rho_o}{\rho}\right)d\psi = (udx + vdy) + i(udy - vdx) = we^{-i\delta}\,dz$$

and the condition

$$\frac{\partial^2 z}{\partial\delta\partial w} = \frac{\partial^2 z}{\partial w\partial\delta}$$

there are obtained the general hodograph equations for a two-dimensional compressible gas

$$\varphi_w = w\left(\frac{\rho_o}{\rho w}\right)_w \psi_\delta$$

$$\varphi_\delta = \left(\frac{\rho_o w}{\rho}\right)\psi_w$$

(c) Hodograph equations for adiabatic flow in terms of

$$\tau = \left(\frac{w}{w_m}\right)^2$$

$$2\tau(1-\tau)^{\frac{\gamma}{\gamma-1}} \quad \varphi_\tau = -\left[1 - \left(\frac{\gamma+1}{\gamma-1}\right)\tau\right] \psi_\delta$$

$$(1-\tau)^{\frac{1}{\gamma-1}} \varphi_\delta = 2\tau \ \psi_\tau$$

(d) Defining expressions for φ and ψ (obtained from equation 1(c))

$$\frac{\partial}{\partial\tau}\left[\frac{\tau}{(1-\tau)^{\frac{1}{\gamma-1}}} \quad \frac{\partial\psi}{\partial\tau}\right] = -\frac{1}{4} \frac{1-\left(\frac{\gamma+1}{\gamma-1}\right)\tau}{\tau(1-\tau)^{\frac{\gamma}{\gamma-1}}} \frac{\partial^2\psi}{\partial\delta^2}$$

$$\frac{\partial}{\partial\tau}\left[\frac{\tau(1-\tau)^{\frac{\gamma}{\gamma-1}}}{1-\left(\frac{\gamma+1}{\gamma-1}\right)\tau}\right] \frac{\partial\varphi}{\partial\tau} = -\frac{1}{4} \frac{(1-\tau)^{\frac{1}{\gamma-1}}}{\tau} \frac{\partial^2\varphi}{\partial\delta^2}$$

3. The von Kàrmàn solution

Simplification of 2(c) by replacing the adiabatic relation

$$\left(\frac{p_1}{p_2}\right) = \left(\frac{\rho_1}{\rho_2}\right)^\gamma$$

by

$$p_1 - p_2 = c\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)$$

where c is chosen so that both relations are tengent at point ρ_s, P_s (free stream) leads to the Von Kármán-Tsien approximating relation

$$P = \frac{P_i}{\sqrt{1-M_s^2} + \dfrac{M_s^2 P_i}{2\left(1+\sqrt{1-M_s^2}\right)}}$$

4. Temple-Yarwood solution

A "best possible approximation" to the solution of the hodograph equations is given by Temple and Yarwood; the relation between P and P_i assumes the form

$$P = \frac{2}{\gamma M_s^2}\left(1+\frac{1}{2}(\gamma-1)M_s^2\left\{1-\left(1-P_i\right)\left[3\left(1-\frac{\frac{1}{4}M_s^2}{1+\frac{1}{5}M_s^2}\right)\frac{\cos\frac{1}{3}(\pi+\epsilon)}{\cos\epsilon}\right]^2\right\}\right)^{\frac{\gamma}{\gamma-1}} - \frac{2}{\gamma M_s^2}$$

where

$$\cos\epsilon = \frac{3}{2}\left(1-\frac{\frac{1}{4}M_s^2}{1+\frac{1}{5}M_s^2}\right)\sqrt{3(1-P_i)\left(\frac{\frac{1}{4}M_s^2}{1+\frac{1}{5}M_s^2}\right)}$$

and

$$0^\circ < \epsilon < 90^\circ$$

It is to be noted that in 3 an approximation to the actual flow conditions is made while 4 approximates, with known accuracy, a mathematical expression of the exact solution.

The results of Temple's paper are presented in several figures, the first of which gives the value of the velocity ratio at any Mach number for a given low-speed (incompressible) velocity ratio.

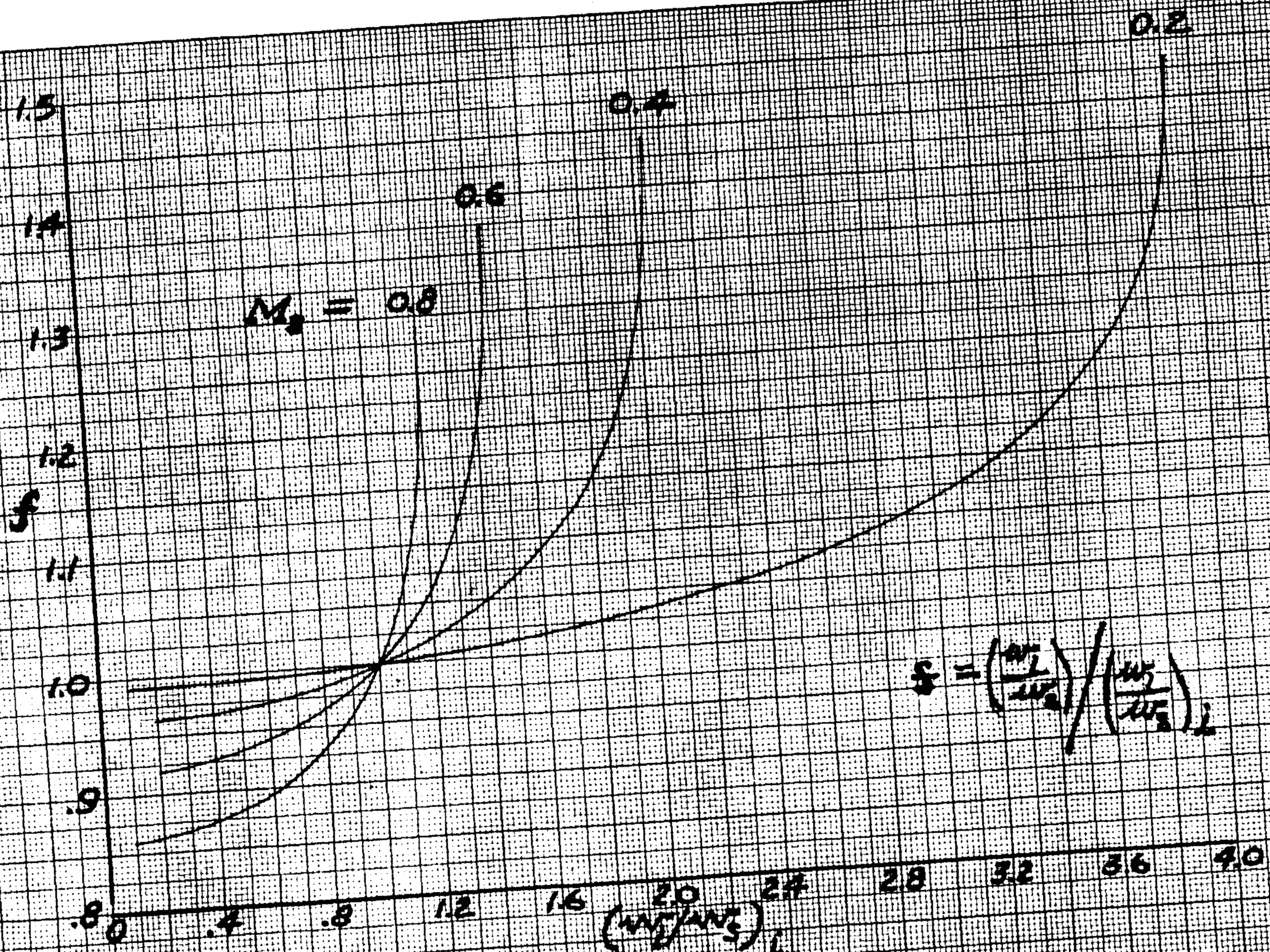


Figure XIh- Velocity correction from Hodograph method.

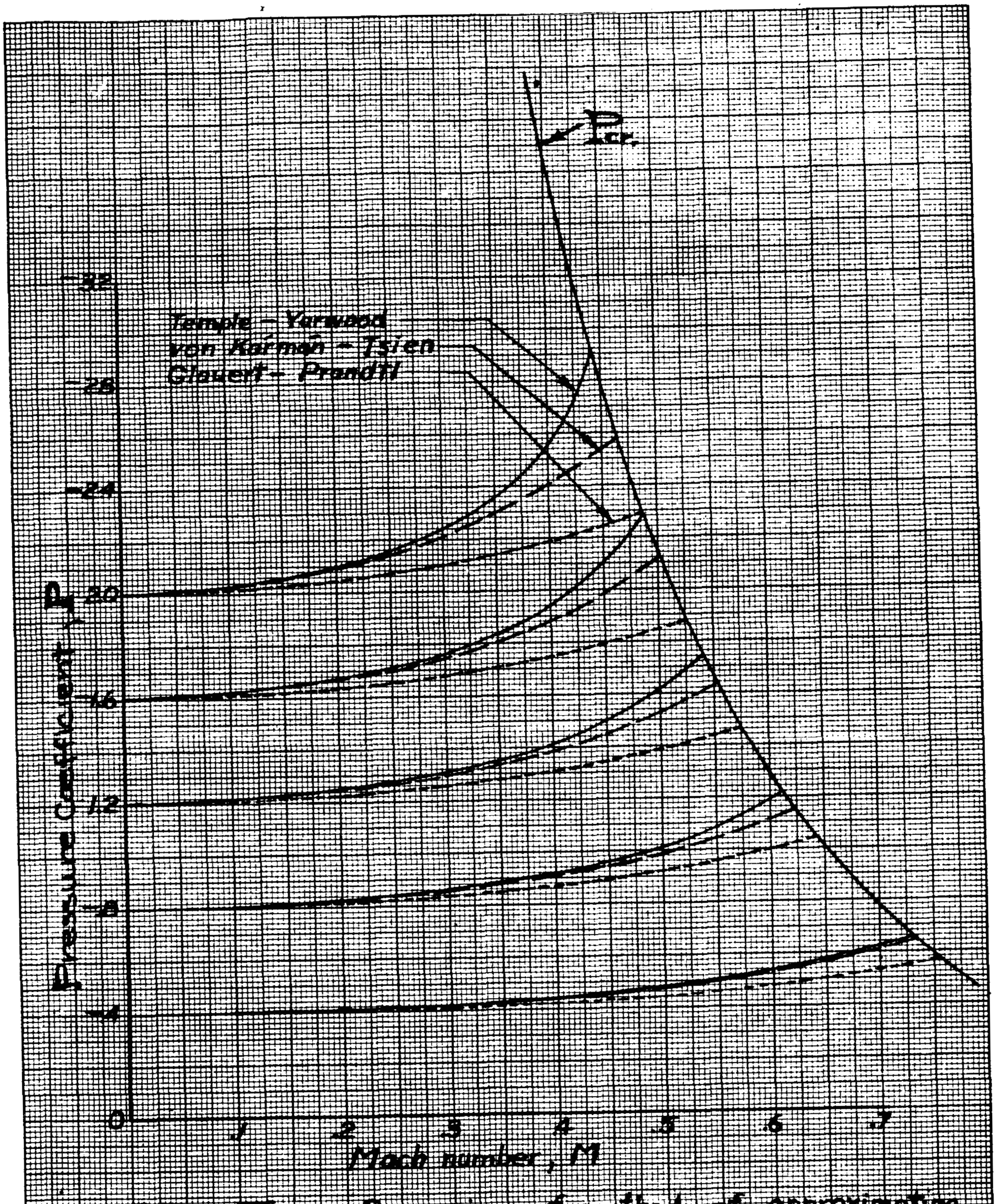


Figure IX c, – Comparison of methods of approximating the change of pressure coefficient with Mach number.

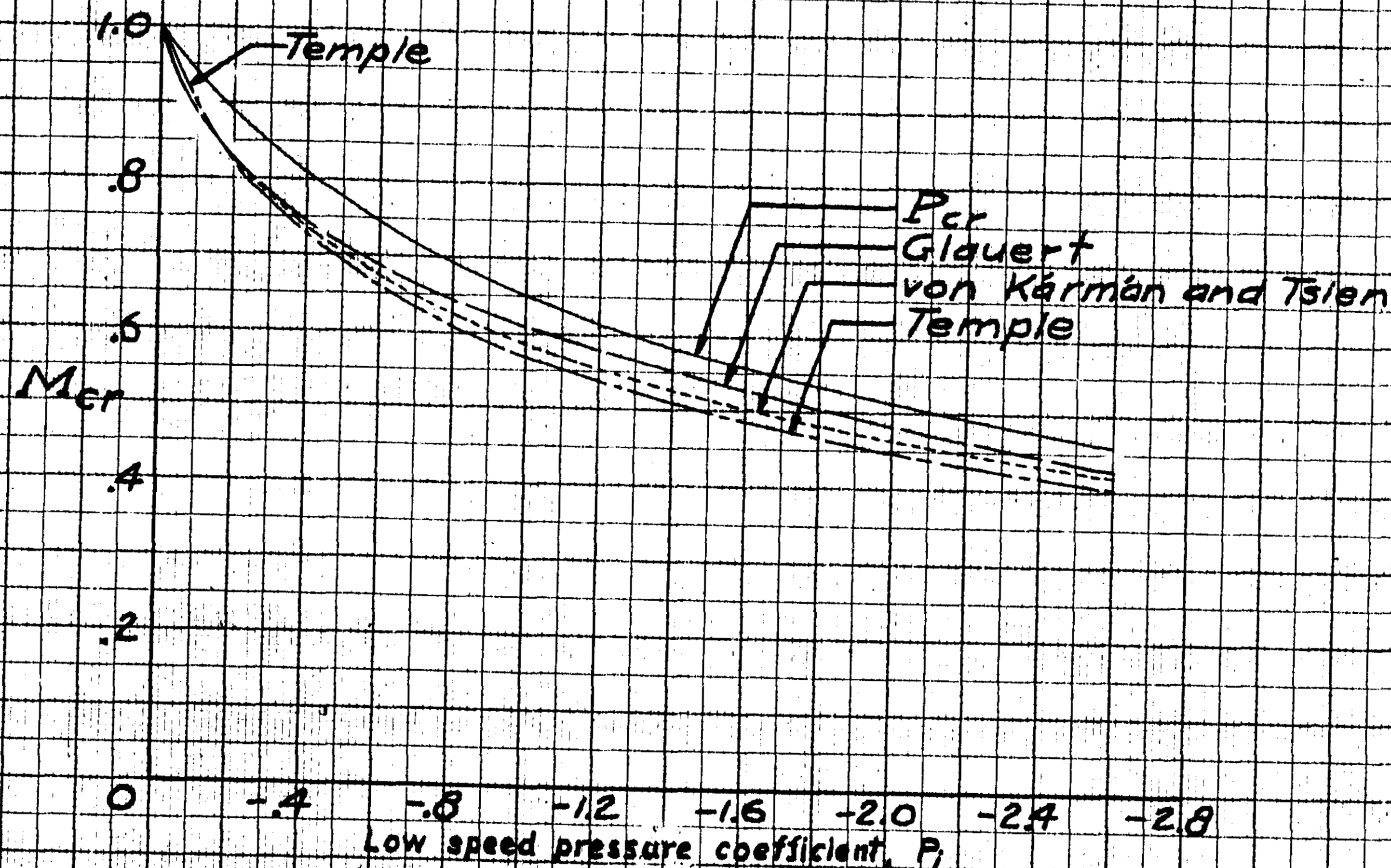


Figure IVd.- Estimation of critical Mach number from low speed pressure coefficient.

X

An Hydraulic Analogy of Two-Dimensional Compressible Gas Flow as Presented by Preiswerk and Others

In this section the enthalpy or heat content h is replaced by $i = C_pT$ and h denotes the water depth at any point; also, C_p is defined for unit weight of fluid and hence g must now be retained.

1. Physical requirements

Smooth water flow with free surface over a horizontal bottom, side boundaries vertical.

2. Energy considerations

(a) Equation for water

$$w^2 = 2g(h_o - h)$$

and

$$w_m = \sqrt{2gh_o}$$

(b) Equation for gas

$$w^2 = 2g(i_o - i) = 2gC_p(T_o - T)$$

and

$$w_m = \sqrt{2gi_o}$$

where C_p and i, are defined for unit weight of fluid

3. Continuity considerations

(a) Equation for water

$$\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0$$

(b) Equation for gas

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Hence for identical expressions

$$\frac{\rho}{\rho_o} = \frac{h}{h_o}$$

also since for adiabatic flow

$$\frac{\rho}{\rho_o} \left(\frac{T}{T_o}\right)^{\frac{1}{\gamma-1}}$$

the above conditions require $\gamma = 2$.

4. The velocity potential

(a) Differential equation for velocity potential of the water flow

$$\varphi_{xx}\left(1 - \frac{\varphi_x{}^2}{gh}\right) + \varphi_{yy}\left(1 - \frac{\varphi_y{}^2}{gh}\right) - 2\varphi_x\varphi_y\frac{\varphi_{xy}}{gh} = 0$$

(b) Differential equation for velocity potential of two-dimensional compressible gas flow

$$\varphi_{xx}\left(1 - \frac{\varphi_x{}^2}{a^2}\right) + \varphi_{yy}\left(1 - \frac{\varphi_y{}^2}{a^2}\right) - 2\varphi_x\varphi_y\frac{\varphi_{xy}}{a^2} = 0$$

5. The complete hydraulic analogy may be summarized as follows:

	Two-dimensional gas flow	Liquid flow with free surface in gravity field
Nature of the flow medium	Hypothetical gas with $\gamma = 2$	Incompressible fluid (e.g. water)
Side boundaries geometrically similar		Side boundary vertical, bottom horizontal
Analogous magnitudes	Velocity $\frac{w}{w_{max}}$, $\frac{w}{a}$	Velocity $\frac{w}{w_{max}}$, $\frac{w}{a}$
	Temp. ratio $\frac{T}{T_o}$	Water depth ratio $\frac{h}{h_o}$
	Density ratio $\frac{\rho}{\rho_o}$	Water depth ratio $\frac{h}{h_o}$
	Pressure ratio $\frac{p}{p_o}$	Square of water depth ratio $\left(\frac{h}{h_o}\right)^2$
	Pressure on the side boundary walls $\frac{p}{p_o}$	Force on the vertical walls $\frac{P}{P_o} = \left(\frac{h}{h_o}\right)^2$
	Sound velocity a	Wave velocity $\sqrt{gh}$
	Mach number $\frac{w}{a}$	Mach number $\frac{w}{\sqrt{gh}}$
	Subsonic flow	Streaming water
	Supersonic flow	Shooting water
	Compressive shock (right and slant)	Hydraulic jump (normal and slant)

An Electric Analogy of Two-Dimensional Compressible Flow

"Through three short notes by Riabouchinsky and Demtchenko in the Comptes Rendus of the Paris Academy, 1932, an old work by Tchaplygin written in the Russian language in 1904 became known, wherein it was shown that, for the case of two-dimensional flow, the problem may, by transformation to new coordinates (rectangular or polar), be presented in such a form that a potential flow between two plates at a predetermined variable distance apart, may be represented by an electric flow in an electrolyte of variable depth. One coordinate χ will be a function of the ratio of the local velocity to the maximum (which may naturally also be written as a function of the ratio of the density to the maximum density); the other coordinate, θ , is the direction angle of the velocity of the flow, the variable distance between the two plates being a function of the first coordinate only."